WWU MÜNSTER

Partitioning ℝ^3 in unit circles

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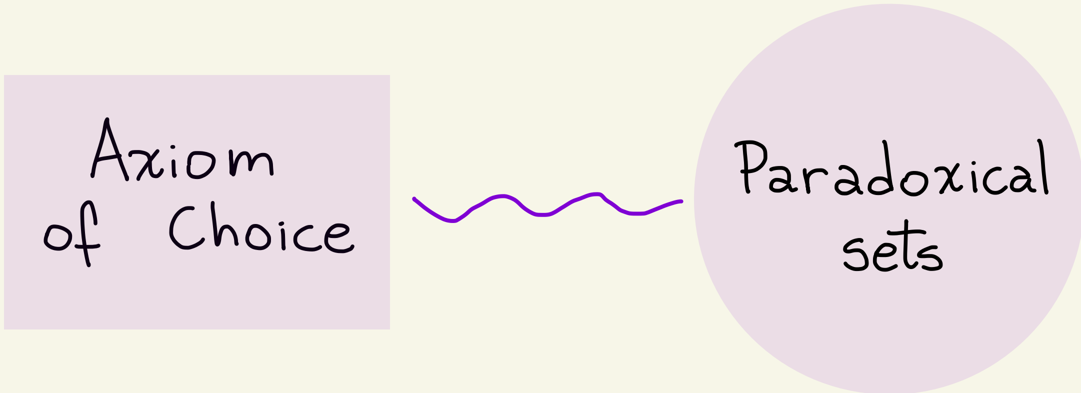
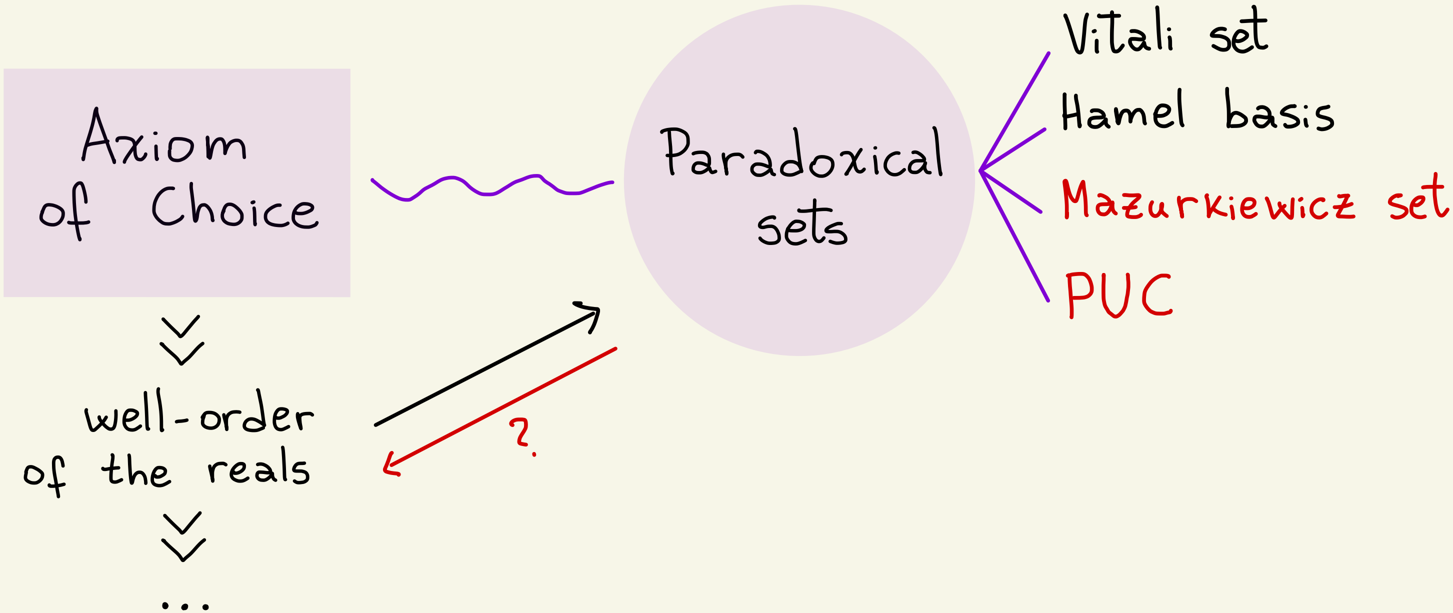
Joint work with Prof. Ralf Schindler

Context



Context


Axiom of Choice
well-order of the reals
...
Paradoxical sets
?
Vitali set
Hamel basis
Mazurkiewicz set
PUC

Partitions in circles

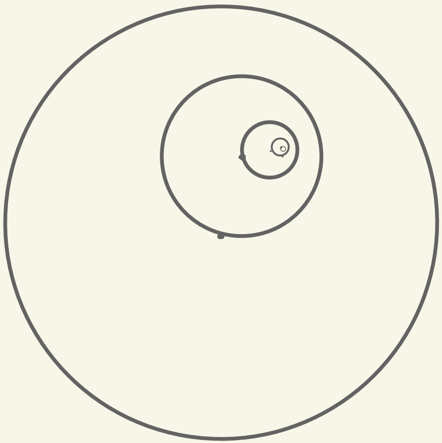
PUC := partition of $\mathbb{R}^3$ in unit circles

Question: (i) Why $\mathbb{R}^3$?

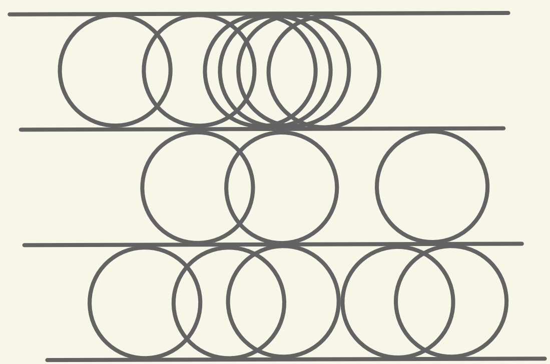
(ii) Why partitions? (1-covering)

(iii) Why circles? Why unit circles?

(i)



(ii)



Partitions in circles

Theorem (ZF) (Szulkin)
$\mathbb{R}^3$ can be partitioned in circles.

Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Partitions in circles

Theorem (ZF) (Szulkin)
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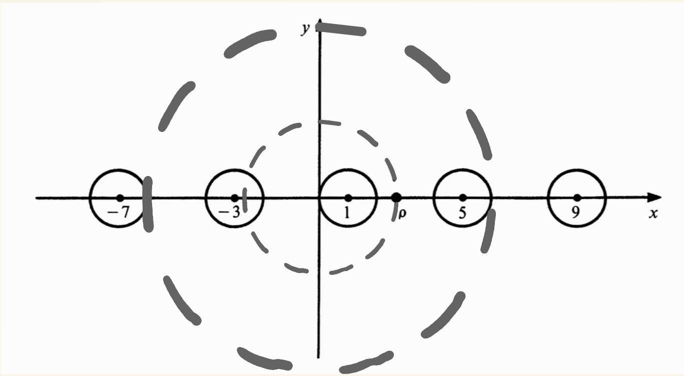
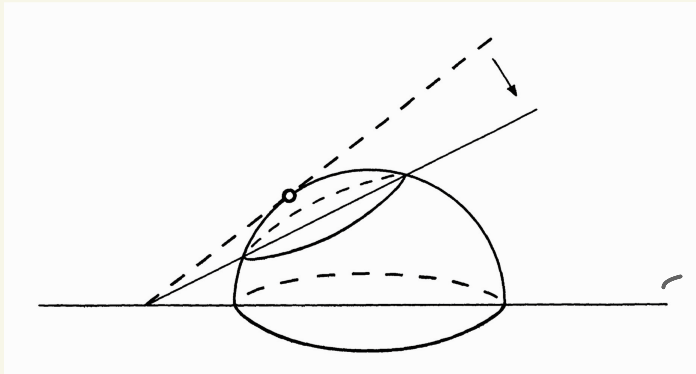


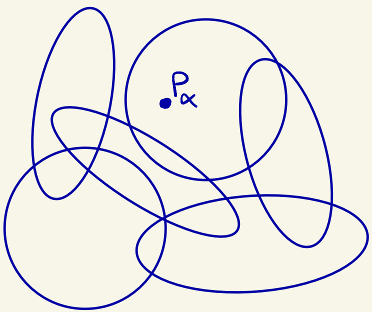


Partitions in unit circles

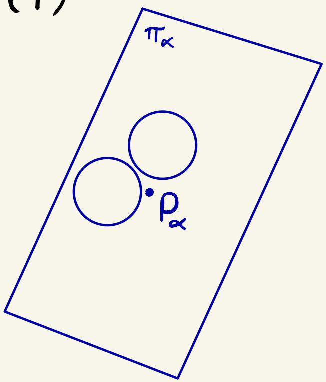
Theorem (ZFC) (Conway-Croft / Kharazishvili)
$\mathbb{R}^3$ can be partitioned in unit circles.

Let $\mathbb{R}^3 = \{p_\alpha\}_{\alpha<\mathfrak{c}}$.

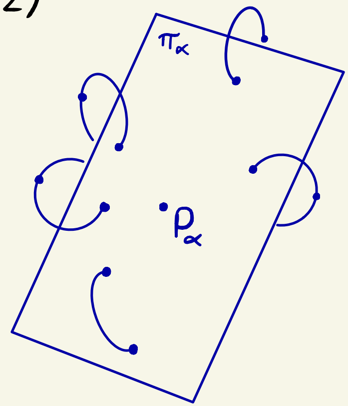
(0)



(1)



(2)



Partitions in unit circles

Observation: The proof shows that any partial PUC of cardinality $< \mathfrak{c}$ can be extended to a (complete) PUC.

Question: Can we always extend a partial PUC to a (complete) PUC?

Partitions in unit circles

Observation: The proof shows that any partial PUC of cardinality $< \mathfrak{c}$ can be extended to a (complete) PUC.

Question: Can we always extend a partial PUC to a (complete) PUC?

- Sometimes there is not enough "space" to extend a partial PUC.

The result

Theorem

There is a model of ZF + no well-order of $\mathbb{R}$ + $\exists$ PUC

The model(s)

1. Cohen-Halpern-Lévy model:

$$H := \mathrm{HOD}^{L[g]}_{A \cup \{A\}}$$

where g is $\mathbb{C}(\omega)$-generic over L, and

$A = \{c_n : n < \omega\}$ is the set of Cohen reals added by g.

2.

$$W = L(\mathbb{R}, b)^{L[\tilde{g}, h]}$$

where $\tilde{g}$ is $\mathbb{C}(\omega_1)$-generic over L,

h is $\mathbb{P}$-generic over $L[\tilde{g}]$, and

$b = \bigcup h$ is the PUC added by h.

Last time!

Cohen-Halpern-Lévy model

1. Cohen-Halpern-Lévy model:

$$H := \mathrm{HOD}^{L[g]}_{A \cup \{A\}}$$

where g is $\mathbb{C}(\omega)$-generic over L, and $A = \{c_n : n < \omega\}$ is the set of Cohen reals added by g.

Facts about H

(i) There is no well-ordering of the reals.

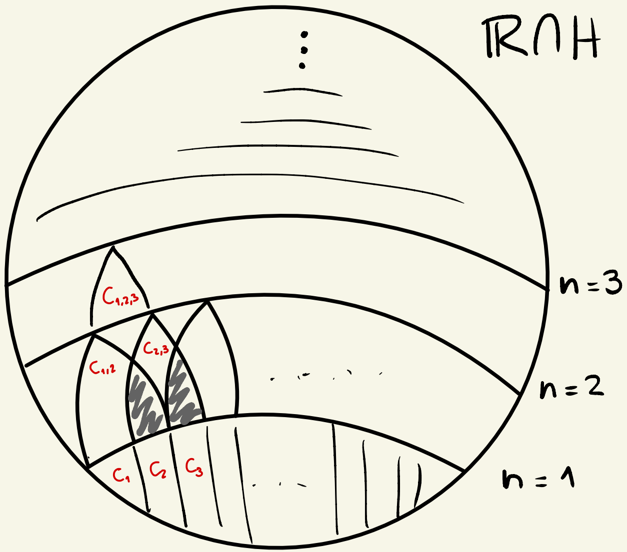
(ii) There is no countable subset of A.

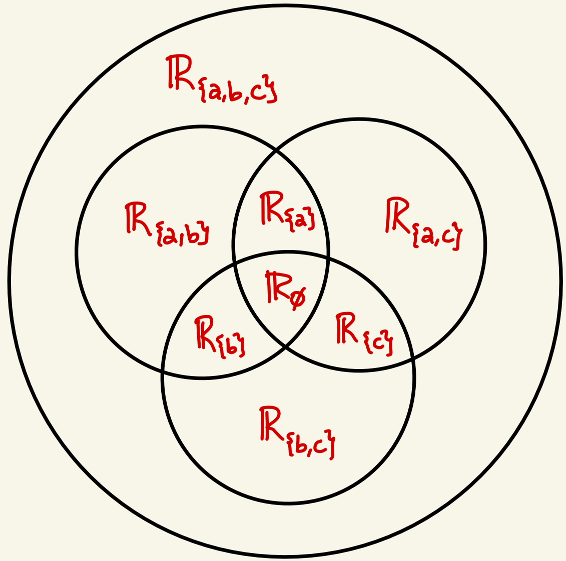
(iii) $\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a])$

Construction of a PUC in H

Q: How do we get a PUC in H?

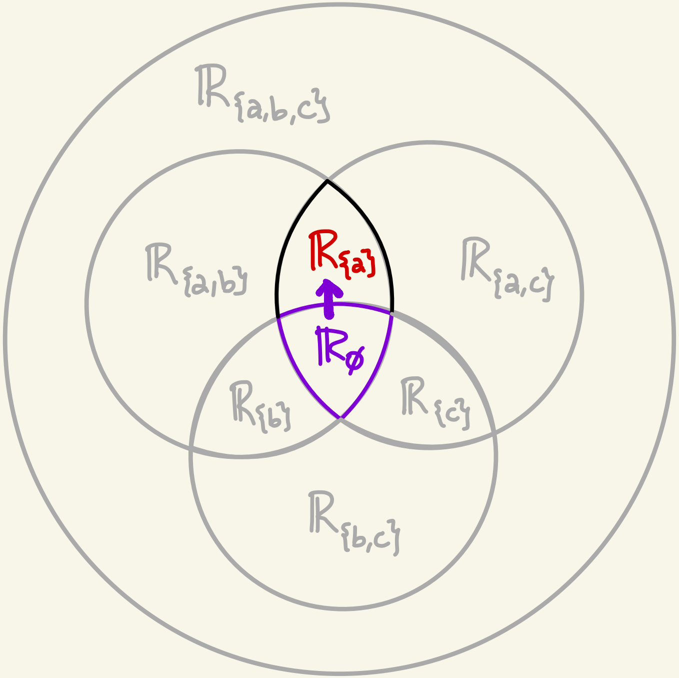
$$\mathbb{R} \cap H = \bigcup_{a \in [A]^{<\omega}} (\mathbb{R} \cap L[a]) = \bigcup_{n<\omega} \bigcup_{\substack{|a|=n \\ a \subseteq A}} \mathbb{R}_a$$



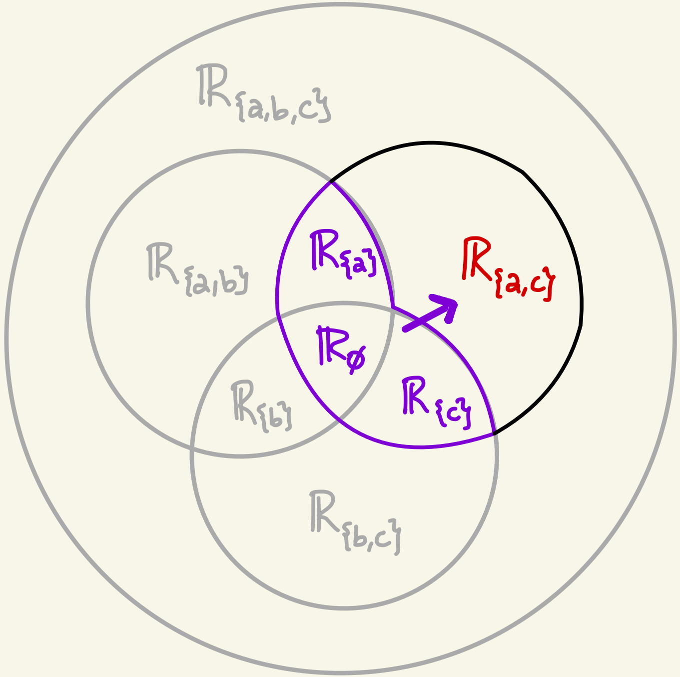


Construction of a PUC in H

The problems that arise



Extendability

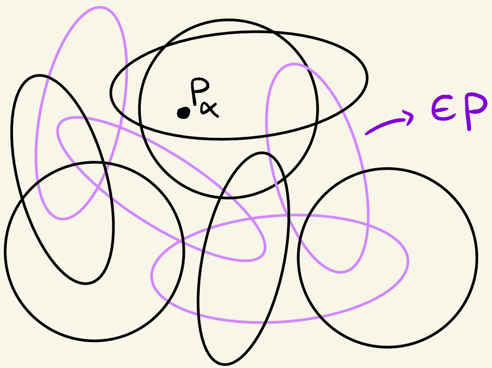


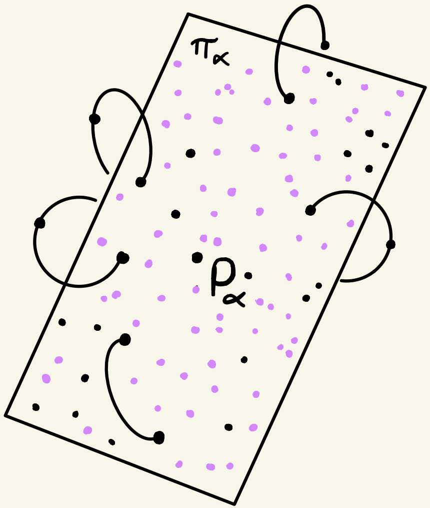
(Strong) Amalgamation

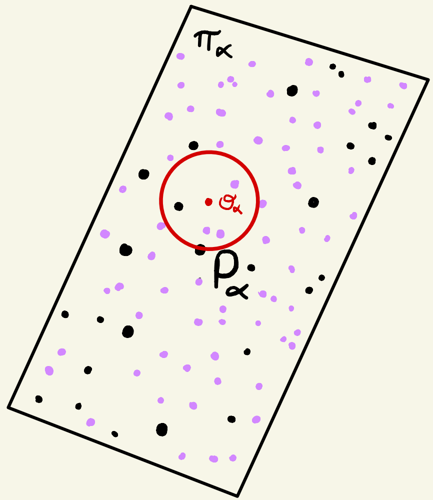
Construction of a PUC in H

Lemma 1 (Extendability)
Let V be a ZFC model and $p \in V$ such that $V \models$ "p is a (partial) PUC". Let c be a Cohen real over V. Then, there is $q \in V[c]$ s.t. $V[c] \models$ "$q \supseteq p \wedge q$ is a PUC"

$$\mathbb{R}^3 \setminus \bigcup p = \{P_\alpha\}_{\alpha < \mathfrak{c}}$$



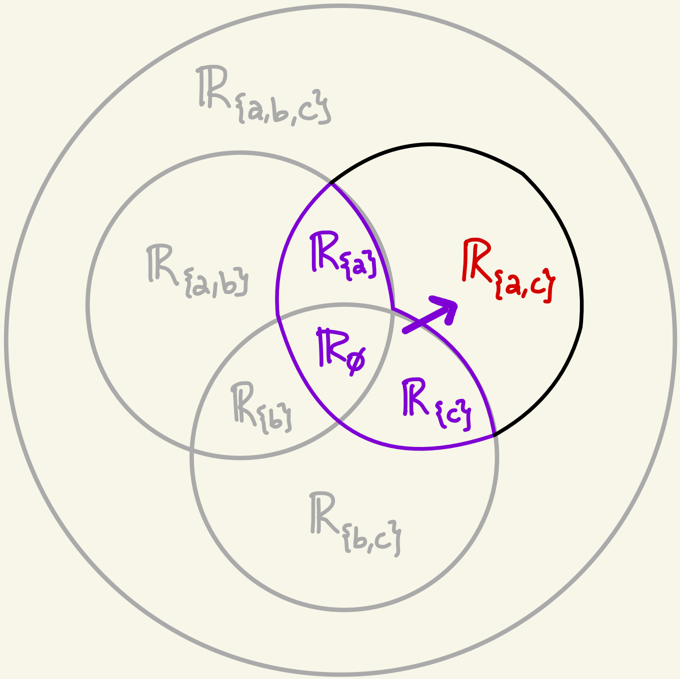




Construction of a PUC in H

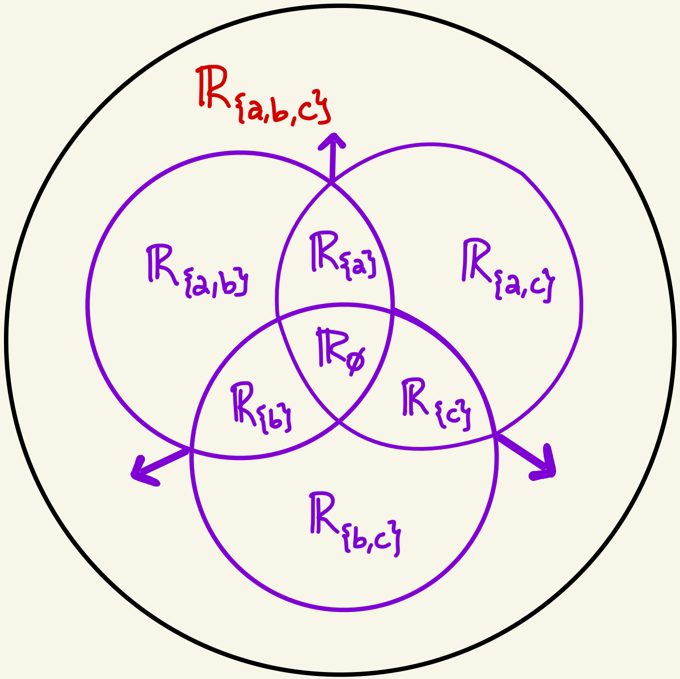
Fact: Let $V \models ZFC$ and $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^{V}$ is $\mathfrak{c}$.

Construction of a PUC in H



(Strong) Amalgamation $(n=2)$

Construction of a PUC in H



(Strong) Amalgamation ($n=3$)

Construction of a PUC in H

Lemma 2 (Strong Amalgamation) $n=2$

Let a, b, c mutually generic Cohen reals and let p, q_1, q_2 be such that

$$\begin{cases} L[a] \models p \text{ is a PUC} \\ L[a,b] \models q_1 \text{ is a PUC} \\ L[a,c] \models q_2 \text{ is a PUC} \end{cases}$$

and $q_1, q_2 \leq_{\mathbb{P}} p$.

Then $L[a,b,c] \models q_1 \cup q_2$ is a partial PUC and it can be extended to a PUC $q \leq_{\mathbb{P}} q_1 \cup q_2$

Algebraic detour

Fact: Let $V \models ZFC$ and $V[c]$ be a generic extension obtained by adding one Cohen real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

Lemma (Transcendence degree)

Let V be a model of ZFC and let S be a finite set of mutually generic Cohen reals.

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over $\overline{\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}}^{alg}$ is $\mathfrak{c}$.

Q: What can we say if the reals are not Cohen reals?

Algebraic detour

Lemma (B. de Bondt)
Let $V \models ZFC$ and $V[c]$ be a generic extension obtained by adding one ~~Cohen~~ real. Then the transcendence degree of $\mathbb{R}^{V[c]}$ over $\mathbb{R}^V$ is $\mathfrak{c}$.

Lemma (Transcendence degree)

Let V be a model of ZFC and let S be a finite set of mutually $\mathbb{P}$-generic reals (*)

Then the transcendence degree of $\mathbb{R}^{V[S]}$ over $\overline{\bigcup_{\substack{T \subseteq S \\ |T| = |S|-1}} \mathbb{R}^{V[T]}}^{alg}$ is $\mathfrak{c}$.

(*) $\mathbb{P}$ from a certain nice family

To be continued…

Thank you for you attention!

References

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